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MULTI-DIMENSIONAL RADIATIVE TRANSFER IN NON-ISOTHERMAL CYLINDRICAL MEDIA WITH NON-ISOTHERMAL BOUNDING WALLS

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Abstract—A systematic approach for the exact calculation of multi-dimensional radiative heat flux in a cylindrical emitting–absorbing non-isothermal medium with non-isothermal bounding walls is described. Closed form exact solutions are obtained for radiative transfer inside (finite and infinite) cylinders and concentric cylinders with prescribed three-dimensional temperature distribution. For the special case of one-dimensional cylindrically symmetric situation, the exact solution obtained in the present work is shown to be equivalent to the exact solution obtained in a previous study, although the solution presented here is much more elegant in approach and simpler in form. Numerical results are also presented for the case of isothermal media bounded by piecewise isothermal walls.

NOMENCLATURE

Bu, $\alpha_a L$ where α_a is the absorption coefficient and L is the reference length;
C, constant of integration defined in equations (1) and (3);
c, c_1, c_2, length of the finite cylinder;
I, specific radiation intensity;
L, reference length;
l_r, l_Φ, l_z, direction cosines defined in equation (5);
M, space-integrated radiation intensity;
Q_r, Q_Φ, Q_z, normalized radiation heat flux in r, Φ, and z-direction;
q_r, q_Φ, q_z, radiation heat flux in r, Φ and z-direction;
$\mathbf{R}$, position vector in the cylindrical coordinate system;
r, radius of the cylinder;
s_1, s_2, s_3, coordinates defined by equation (2) for a three-dimensional temperature field, and by equation (4) for a two-dimensional temperature field;
T, temperature;
z, one of the cylindrical coordinates.

Greek symbols

α, angle defined by equation (42a);
α_a, absorption coefficient;
β, angle defined by equation (42b);
θ, angle defined in Fig. 1;
$\theta_B, \theta_C, \theta_{BC}, \theta_E, \theta_F$, angles defined by equations (17f), (17g), (21b), (25b), and (25c);
σ, Stefan–Boltzmann constant;
ϕ, angle defined in Fig. 1;
Φ, angle defined in Fig. 1;
$\phi_A, \phi_B, \phi_C, \phi_D$, angles defined by equations (22b), (22c), (22d) and (25d).

Superscripts

$*$, quantities at the wall;
i, quantities associated with inner cylinder;
o, quantities associated with outer cylinder.

Subscripts

i, quantities associated with inner cylinder;
o, quantities associated with outer cylinder;
g, radiative quantities resulting from medium emission;
w, radiative quantities resulting from wall emission.

INTRODUCTION

THE CALCULATION of radiative transfer in a cylindrical non-isothermal emitting–absorbing medium has attracted considerable attention in the past decade, because of its important applications in many high-temperature phenomena associated with planetary reentry, nuclear explosion, laboratory shock-tube studies, nuclear reactor, and industrial furnace designs. Heaslet and Warming [1] as well as Kestin [2] have studied the problem of radiative transfer in a non-isothermal medium inside an axial symmetric infinite cylinder. Tien and Abu-Romia [3] has obtained an exact solution for radiative transfer outside of a semi-infinite isothermal cylindrical medium, while Desoto [4] considered the problem of emitting–absorbing axial symmetric pipe flow. These exact solutions for radiative

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transfer in cylindrically symmetric medium are suggestive of having been obtained entirely from geometric considerations with no systematic and rigorously analytical approach to the problem; hence there is little scope to take into account the three-dimensional effects.

The difficulty in exact calculation of multi-dimensional radiative transfer has prompted researchers to develop approximate schemes such as the mean-beam length method [5, 6], the zoning method [6–8], the differential approximation [9–11], and the Monte Carlo method [5, 12, 13] among others. Of all of these techniques, the differential approximation appears to be the most convenient, when applied to problems in radiation-coupled flows; because of its simplicity, generality, and analytical nature. However, the differential approximation has its own shortcomings. A comparison of the exact solution and differential approximation for the simple case of cylindrically symmetric one-dimensional situation shows that the differential approximation over-estimates the radiative heat flux on the inner surface of the concentric cylinders by a factor of two in the thin-gas limit, when the radius of the inner surface is small compared with the outer one [14]. Moreover, recent studies [15] suggest that the differential approximation would breakdown wherever the average radiation intensity is discontinuous; a situation that will arise where the wall temperature is discontinuous. Thus an analytical approach for the exact calculation of multi-dimensional radiative transfer in cylindrical media is of fundamental and practical interest.

A systematic and rigorous approach for the exact calculation of multi-dimensional radiative heat flux in rectangular geometries has been recently advanced by Cheng [15]. In this paper, we shall extend this method to problems in cylindrical configurations. Particular attention will be placed on problems with discontinuous wall temperature distribution.

EXACT SOLUTIONS

In this section, we shall obtain the exact solutions of the radiation-transport equations in cylindrical coordinate system (see Fig. 1) when the temperature distribution of the medium and the wall are prescribed. We shall show that:

(i) for a three-dimensional temperature field (r, Φ, z) as well as axial symmetric temperature field (r, z), the formal solution to the radiation-transport equation is

$$I(s_1, s_2, s_3, \theta, \phi) = C(s_2, s_3, \phi, \theta)\exp[-\alpha_a s_1] + \frac{\sigma}{\pi}\int_0^{s_1 - s_1*} \alpha_a T_g^4(\bar{s}_1', s_2, s_3, \phi, \theta) \times \exp[\alpha_a(\bar{s}_1' - \bar{s}_1)]\, d\bar{s}_1', \quad (1)$$

where

$$s_1 = r\sin\theta\cos\phi + z\cos\theta, \quad s_2 = -r\sin\phi,$$
$$s_3 = r\cos\theta\cos\phi - z\sin\theta,$$
$$s_1^* = r^*\sin\theta^*\cos\phi^* + z^*\cos\theta^* \text{ and } \bar{s}_1' = s_1' - s_1^*, \quad (2)$$

(ii) for two-dimensional (r, Φ) and one-dimensional (r) temperature fields the formal solution for radiation intensity is

$$I(s_1, s_2, \phi, \theta) = C(s_2, \phi)\exp\left[-\frac{\alpha_a s_1}{\sin\theta}\right] + \frac{\sigma}{\pi}\int_0^{(s_1 - s_1^*)/\sin\theta} \alpha_a T_g^4(\bar{s}_1', s_2, \phi, \theta) \times \exp[\alpha_a(\bar{s}_1' - \bar{s}_1)]\, d\bar{s}_1', \quad (3)$$

where

$$s_1 = r\cos\phi, \quad s_2 = -r\sin\phi,$$
$$s_1^* = r^*\cos\phi^*, \quad \text{and} \quad \bar{s}_1' = (s_1' - s_1^*)/\sin\theta, \quad (4)$$

with the superscript "*" denoting quantities at the wall. The dummy variable $\bar{s}_1'$ in equations (1) and (3) is the physical distance along s_1 from an arbitrary field point to the bounding wall; and C is to be determined from the boundary condition. For the convenience of discussion, we shall henceforth refer to the first terms in equations (1) and (3) by I_w representing the contribution resulting from wall emission, and the second terms by I_g representing the contribution from medium emission. When the solution for I is obtained, the space integrated radiation intensity and the radiative heat flux are given by

$$\begin{aligned} M(r, \Phi, z) &= \int_\Omega I(r, \Phi, z, \phi, \theta)\, d\Omega, \\ q_r(r, \Phi, z) &= \int_\Omega I(r, \Phi, z, \phi, \theta) l_r\, d\Omega, \\ q_\Phi(r, \Phi, z) &= \int_\Omega I(r, \Phi, z, \phi, \theta) l_\Phi\, d\Omega, \\ q_z(r, \Phi, z) &= \int_\Omega I(r, \Phi, z, \phi, \theta) l_z\, d\Omega, \end{aligned} \quad (5)$$

where $l_r \equiv \mathbf{e}_\Omega \cdot \mathbf{e}_r = \sin\theta\cos\phi$, $l_\Phi \equiv \mathbf{e}_\Omega \cdot \mathbf{e}_\Phi = \sin\theta\sin\phi$, and $l_z \equiv \mathbf{e}_\Omega \cdot \mathbf{e}_z = \cos\theta$, with $\mathbf{e}_\Omega$ given by

$$\mathbf{e}_\Omega \equiv \sin\theta\cos\phi\,\mathbf{e}_r + \sin\theta\sin\phi\,\mathbf{e}_\Phi + \cos\theta\,\mathbf{e}_z. \quad (6)$$

Three-dimensional temperature field

Consider the problem of radiative transfer in an emitting–absorbing cylindrical medium where the temperature distribution depends on r, Φ, and z. The radiation intensity I is a function of position vector $\mathbf{R}$ given by $\mathbf{R} = r\,\mathbf{e}_r + z\,\mathbf{e}_z$, and the unit directional vector $\mathbf{e}_\Omega$.

The radiation-transport equation for an emitting–absorbing grey medium in local thermodynamic equilibrium is given by [16]

$$\left[\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\Phi}-\frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\cos\theta\frac{\partial}{\partial z}+\alpha_a\right]I=\alpha_a\sigma T_g^4(r,\Phi,z)/\pi, \quad (7)$$

where α_a is the absorption coefficient and σ the Stefan–Boltzmann constant. If a black bounding wall with temperature $T_w(r^*,\Phi^*,z^*)$ exists in the radiation field, the radiative boundary condition is given by

$$I(r^*,\Phi^*,z^*,\phi^*,\theta^*)=\sigma T_w^4(r^*,\Phi^*,z^*)/\pi, \quad (8)$$

where r^*, Φ^*, z^*, ϕ^*, and θ^* are specified.

To obtain the formal solution of equation (7) with boundary condition (8), we recast these equations in terms of the new independent variables $s_j (j = 1,2,3)$ where $s_1 \equiv \mathbf{R}.\mathbf{e}_\Omega$, $s_2 \equiv \mathbf{R}.\mathbf{e}_\phi$, $s_3 \equiv \mathbf{R}.\mathbf{e}_\theta$; with $\mathbf{e}_\Omega$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ denoting the local spherical coordinates given by

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_r+\cos\phi\,\mathbf{e}_\Phi,$$
$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{e}_r+\cos\theta\sin\phi\,\mathbf{e}_\Phi-\sin\theta\,\mathbf{e}_z. \quad (9)$$

The explicit expressions for s_1, s_2, s_3 are given in equation (3).

Equations (7) and (8) in terms of the new independent variables are

$$\left[\frac{\partial}{\partial s_1}+\alpha_a\right]I(s_1,s_2,s_3,\theta,\phi)=\frac{\alpha_a\sigma T_g^4}{\pi}(s_1,s_2,s_3,\theta,\phi), \quad (10)$$

$$I(s_1^*,s_2^*,s_3^*,\theta^*,\phi^*)=\frac{\sigma T_w^4}{\pi}(s_1^*,s_2^*,s_3^*,\theta^*,\phi^*), \quad (11)$$

where

$$s_1^* = r^*\sin\theta^*\cos\phi^*+z^*\cos\theta^*, \quad s_2^* = -r^*\sin\phi^*,$$

and

$$s_3^* = r^*\cos\theta^*\cos\phi^*-z^*\sin\theta^*. \quad (12)$$

The solution of equation (10) with boundary condition (11) is given by equation (1). To determine C in equation (1) for each of the boundary conditions, we note from geometric considerations that

$$\theta = \theta^*, \quad (13a)$$
$$\phi^* = 2\pi+(\Phi+\phi-\Phi^*), \quad (13b)$$
$$s_2 = s_2^*, \quad (13c)$$
$$s_3 = s_3^*, \quad (13d)$$

which can be rewritten to give

$$\Phi^* = (2\pi+\Phi+\phi)-\sin^{-1}\left[\frac{r}{r^*}\sin\phi\right], \quad (14a)$$

$$\cos\phi^* = \pm\frac{1}{r^*}\sqrt{(r^{*2}-r^2\sin^2\phi)}, \quad (14b)$$

$$z^* = z-\cot\theta(r\cos\phi-r^*\cos\phi^*). \quad (14c)$$

It is interesting to note that the coordinate Φ appears in the solution through equation (14a).

We shall now discuss the determination of C for some specific problems.

Emitting–absorbing medium inside a finite cylinder. Consider an emitting–absorbing medium with prescribed temperature $T_g(r,\Phi,z)$ inside a finite cylinder with wall temperatures given by $T_o(r_o, \Phi^*, z^*)$, $T_1(r^*,\Phi^*,c_1)$, and $T_2(r^*,\Phi^*,c_2)$ where $c_1 < c_2$. The radiative boundary conditions are given by

$$I(r_o,\Phi^*,z^*,\theta^*,\phi^*)=\frac{\sigma T_0^4}{\pi}(r_o,\Phi^*,z^*),$$
$$c_1 \leqslant z^* \leqslant c_2, \quad \mathbf{e}_r^*.\mathbf{e}_\Omega^* \leqslant 0, \quad (15a)$$
$$I(r^*,\Phi^*,c_1,\theta^*,\phi^*)=\frac{\sigma T_1^4}{\pi}(r^*,\Phi^*,c_1),$$
$$0 \leqslant r^* \leqslant r_o, \quad \mathbf{e}_z^*.\mathbf{e}_\Omega^* \geqslant 0, \quad (15b)$$
$$I(r^*,\Phi^*,c_2,\theta^*,\phi^*)=\frac{\sigma T_2^4}{\pi}(r^*,\phi^*,c_2),$$
$$0 \leqslant r^* \leqslant r_o, \quad \mathbf{e}_z^*.\mathbf{e}_\Omega^* \leqslant 0. \quad (15c)$$

Consider the boundary condition (15a). The limit $\mathbf{e}_r.\mathbf{e}_\Omega \leqslant 0$ implies $\sin\theta^*\cos\phi^* \leqslant 0$ which gives $\cos\phi^* \leqslant 0$ for $0 \leqslant \theta^* \leqslant \pi$. Thus we choose the negative sign for equation (14b) when it is substituted into equation (14c) with $r^* = r_o$ to give

$$z^* = z-\cot\theta[r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}]. \quad (16)$$

With the aid of equations (2), (13), (15a), (15b) and (15c), it can be shown that for $0 \leqslant \phi \leqslant 2\pi$ and $\theta_B \leqslant \theta \leqslant \theta_c$, the medium and wall emissions are given by

$$I_g^{(o)} = \frac{\sigma}{\pi}\int_0^{s_1-s_1^*}\alpha_a T_g^4(\tilde{s}_1', s_2, s_3, \phi, \theta)\times\exp[\alpha_a(\tilde{s}_1'-\tilde{s}_1)]\,\mathrm{d}\tilde{s}_1', \quad (17a)$$

and

$$I_w^{(o)} = \frac{\sigma T_o^4}{\pi}[r_o,\Phi^*,z^*]\exp[-\alpha_a(s_1-s_1^*)], \quad (17b)$$

where

$$s_1-s_1^* = \frac{r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}}{\sin\theta}, \quad (17c)$$

$$\Phi^* = \Phi+\phi-\sin^{-1}\left(\frac{r}{r_o}\sin\phi\right), \quad (17d)$$

$$z^* = z-\cot\theta[r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}], \quad (17e)$$

$$\theta_B \equiv \tan^{-1}\left\{\frac{r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}}{z-c_1}\right\}, \quad (17f)$$

$$\theta_C \equiv \tan^{-1}\left\{\frac{r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}}{z-c_2}\right\}. \quad (17g)$$

The limits for θ, in which equations (17a) and (17b) are valid, follow from the condition $c_2 \leqslant z^* \leqslant c_1$ with the aid of equation (17e).

Similarly, imposing boundary condition (15b) gives

$$I_g^{(1)} = \frac{\sigma}{\pi}\int_0^{s_1 - s_1^*} \alpha_a T_g^4[\bar{s}_1', s_2, s_3, \phi, \theta] \times \exp[\alpha_a(\bar{s}_1' - \bar{s}_1)]\,d\bar{s}_1', \quad (18a)$$

$$I_w^{(1)} = \frac{\sigma T_1^4}{\pi}[r^*, \Phi^*, c_1]\exp[-\alpha_a(s_1 - s_1^*)], \quad (18b)$$

where

$$s_1 - s_1^* = \frac{z - c_1}{\cos\theta}, \quad (18c)$$

$$r^* = \sqrt{\{r^2\sin^2\phi + [(z-c_1)\tan\theta - r\cos\phi]^2\}}, \quad (18d)$$

and

$$\Phi^* = \Phi + \phi - \sin^{-1}\left\{\frac{r\sin\phi}{\sqrt{\{r^2\sin^2\phi + [(z-c_1)\tan\theta - r\cos\phi]^2\}}}\right\}. \quad (18e)$$

Equations (18a) and (18b) are valid in the region $0 \leqslant \theta \leqslant \theta_B$ and $0 \leqslant \phi \leqslant 2\pi$. Similarly, imposing boundary condition (15c), we obtain the expressions for $I_g^{(2)}$ and $I_w^{(2)}$ which are identical in form as equations (18a) and (18b) with $T_1[r^*, \Phi^*, c_1]$ replaced by $T_2[r^*, \Phi^*, c_2]$ and with equations (18c), (18d) and (18e) replaced by

$$s_1 - s_1^* = \frac{z - c_2}{\cos\theta}, \quad (19a)$$

$$r^* = \sqrt{\{r^2\sin^2\phi + [(z-c_2)\tan\theta - r\cos\phi]^2\}}, \quad (19b)$$

$$\Phi^* = \Phi + \phi - \sin^{-1}\left\{\frac{r\sin\phi}{\sqrt{\{r^2\sin^2\phi + [(z-c_2)\tan\theta - r\cos\phi]^2\}}}\right\}. \quad (19c)$$

The expressions for $I_g^{(2)}$ and $I_w^{(2)}$ are valid in the limits $\theta_c \leqslant \theta \leqslant \pi$, and $0 \leqslant \phi \leqslant 2\pi$.

Thus the radiative quantities resulting from medium emission inside a finite cylinder is given by

$$\begin{Bmatrix} M_g(r,\Phi,z) \\ q_{gr}(r,\Phi,z) \\ q_{g\Phi}(r,\Phi,z) \\ q_{gz}(r,\Phi,z) \end{Bmatrix} = \int_0^{2\pi}\int_0^{\theta_B} I_g^{(1)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_B}^{\theta_C} I_g^{(o)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_C}^{\pi} I_g^{(2)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega, \quad (20)$$

and those resulting from wall emission are also given by equation (20) with $I_g^{(o)}$, $I_g^{(1)}$, and $I_g^{(2)}$ replaced by $I_w^{(o)}$ $I_w^{(1)}$, and $I_w^{(2)}$.

We now consider the cases when the wall temperature is discontinuous. In these cases, the expressions for medium emission remains the same while the wall emission must be modified as follows.

(i) If wall temperature at $r = r_o$ is discontinuous at $z = 0$ with $T_{c+}[r_o, \Phi^*, z^*]$ for $0 < z^* \leqslant c_2$ and $T_{c-}[r_o, \Phi^*, z^*]$ for $c_1 \leqslant z^* < 0$, the radiative quantities resulting from wall emission are given by

$$\begin{Bmatrix} M_w(r,\Phi,z) \\ q_{wr}(r,\Phi,z) \\ q_{w\Phi}(r,\Phi,z) \\ q_{wz}(r,\Phi,z) \end{Bmatrix} = \int_0^{2\pi}\int_0^{\theta_B} I_w^{(1)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_B}^{\theta_{BC}} I_w^{(c-)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_{BC}}^{\theta_C} I_w^{(c+)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_C}^{\pi} I_w^{(2)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega, \quad (21a)$$

where $I_w^{(c\pm)}$ is given by equation (17b) with T_o replaced by $T_{c\pm}$ and

$$\theta_{BC} \equiv \tan^{-1}\left[\frac{r\cos\phi + \sqrt{(r_o^2 - r^2\sin^2\phi)}}{z}\right]. \quad (21b)$$

(ii) If the wall temperature on $r = r_o$ is discontinuous along the peripheral angle at $\Phi = 0$, with $T_{o+}(r_o, \Phi^*, z^*)$ for $0 < \Phi^* \leqslant \Phi_2$ and $T_{o-}(r_o, \Phi^*, z^*)$ for $\Phi_1 \leqslant \Phi^* < 0$ and zero elsewhere, the radiative quantities resulting from wall emission are

$$\begin{Bmatrix} M_w(r,\Phi,z) \\ q_{wr}(r,\Phi,z) \\ q_{w\Phi}(r,\Phi,z) \\ q_{wz}(r,\Phi,z) \end{Bmatrix} = \int_0^{2\pi}\int_0^{\theta_B} I_w^{(1)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_{\phi_A}^{\phi_B}\int_{\theta_B}^{\theta_C} I_w^{(o-)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega$$

$$+ \int_{\phi_B}^{\phi_C}\int_{\theta_B}^{\theta_C} I_w^{(o+)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega + \int_0^{2\pi}\int_{\theta_C}^{\pi} I_w^{(2)} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix} d\Omega, \quad (22a)$$

where $I_w^{(c\pm)}$ are given by equation (17b) with T_o replaced by $T_{o\pm}$ and

$$\phi_A \equiv \tan^{-1}\left[\frac{r_o \sin(\Phi-\Phi_1)}{r-r_o\cos(\Phi-\Phi_1)}\right], \tag{22b}$$

$$\phi_B \equiv \tan^{-1}\left[\frac{r_o \sin\Phi}{r-r_o\cos\Phi}\right], \tag{22c}$$

$$\phi_C \equiv \tan^{-1}\left[\frac{r_o \sin(\Phi-\Phi_2)}{r-r_o\cos(\Phi-\Phi_2)}\right]. \tag{22d}$$

Axial-symmetric temperature field

Consider radiative transfer in a cylindrical medium with axial-symmetric temperature distribution; i.e. both the medium temperature and wall temperature independent of Φ. The radiation-transport equation and boundary condition are given by

$$\left[\sin\theta\cos\phi\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{\partial z}-\frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\phi}+\alpha_a\right]I = \alpha_a\sigma T_g^4(r,z)/\pi, \tag{23}$$

and

$$I(r^*,z^*,\phi^*,\theta^*) = \frac{\sigma T_w^4}{\pi}(r^*,z^*), \tag{24}$$

where r^*, z^* are the coordinates at the wall.

It can be shown that equations (23) and (24), when expressed in the new independent variables $s_j (j = 1, 2, 3)$ given by equation (2), are identical to equations (10) and (11) which have the formal solution given by (1). To determine the constant C, we note that equations (13a), (13c), (13d) hold also for the axial symmetric situation which in turn gives equations (14b) and (14c). For the case of finite cylinder with axial symmetric temperature distribution, it can be shown that the exact solutions are identical in form with equations (17–22), by neglecting the temperature dependence of Φ^*. That the solutions for the axial symmetric case are directly obtainable from the three-dimensional situation, by just neglecting Φ^*, can be explained by the fact that wall contribution remains the same no matter at what Φ^* the $\mathbf{e}_\Omega$ vector projected backwards meets the wall. This identical nature of two solutions (the general and the axial symmetric) is well anticipated in view of s_j in three-dimensional case being independent of Φ; and hence the physical distance $(s_1-s_1^*)$ of the given point from the wall in the direction of $\mathbf{e}_\Omega$ also is independent of Φ.

Emitting–absorbing medium between concentric cylinders with finite length. Consider an emitting–absorbing medium with axial symmetric temperature distribution inside concentric cylinders with finite length. The outer and inner wall temperatures are given by $T_o(r_o,z^*)$ and $T_i(r_i,z^*)$ respectively. The wall temperatures at $z = c_1$ and $z = c_2$ are given by $T_1(r^*,c_1)$ and $T_2(r^*,c_2)$ respectively. In this case, it can be shown that the radiative quantities are given by

$$\begin{Bmatrix} M_g(r,z) \\ q_{gr}(r,z) \\ q_{gz}(r,z)\end{Bmatrix} = \int_{\phi_D}^{2\pi-\phi_D}\int_0^{\theta_B} I_g^{(1)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega + \int_{\phi_D}^{2\pi-\phi_D}\int_{\theta_B}^{\theta_C} I_g^{(o)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega + \int_{\phi_D}^{2\pi-\phi_D}\int_{\theta_C}^{\pi} I_g^{(2)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega$$
$$+\int_{-\phi_D}^{\phi_D}\int_0^{\theta_E} I_g^{(1)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega + \int_{-\phi_D}^{\phi_D}\int_{\theta_E}^{\theta_F} I_g^{(i)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega + \int_{-\phi_D}^{\phi_D}\int_{\theta_F}^{\pi} I_g^{(2)}\begin{Bmatrix}1\\ l_r\\ l_z\end{Bmatrix}d\Omega \tag{25a}$$

where θ_B, θ_C, $I_g^{(o)}$, $I_g^{(1)}$, and $I_g^{(2)}$ are given by equations (17f), (17g), (17a) and (18a), and

$$\theta_E \equiv \tan^{-1}\left[\frac{r\cos\phi-\sqrt{(r_i^2-r^2\sin^2\phi)}}{z-c_1}\right], \tag{25b}$$

$$\theta_F \equiv \tan^{-1}\left[\frac{r\cos\phi-\sqrt{(r_i^2-r^2\sin^2\phi)}}{z-c_2}\right], \tag{25c}$$

$$\phi_D \equiv \sin^{-1}\left(\frac{r_i}{r}\right), \tag{25d}$$

$$I_g^{(i)} \equiv \frac{\sigma}{\pi}\int_0^{[r\cos\phi-\sqrt{(r_i^2-r^2\sin^2\phi)}]/\sin\theta} \alpha_a T_g^4(\tilde{s}_1', s_2, s_3, \phi, \theta)\exp[\alpha_a(\tilde{s}_1'-\tilde{s}_1)]\,d\tilde{s}_1'. \tag{25e}$$

The radiative quantities resulting from wall emission is also given by equation (25a) with $I_g^{(1)}$ and $I_g^{(2)}$ replaced by $I_w^{(1)}$ and $I_w^{(2)}$, and $I_g^{(i)}$ and $I_g^{(o)}$ replaced by $I_w^{(i)}$ and $I_w^{(o)}$ where

$$I_w^{(i)} = \frac{\sigma T_i^4}{\pi}[r_i, z^*]\exp[-\alpha_a(s_1 - s_1^*)], \tag{26a}$$

with

$$z^* \equiv z - \cot\theta[r\cos\phi - \sqrt{(r_i^2 - r^2\sin^2\phi)}] \quad \text{and} \quad s_1 - s_1^* = r\cos\phi - \sqrt{(r_i^2 - r^2\sin^2\phi)},$$

and

$$I_w^{(o)} = \frac{\sigma T_o^4}{\pi}[r_o, z^*]\exp[-\alpha_a(s_1 - s_1^*)], \tag{26b}$$

where z^*, $s_1 - s_1^*$ are given by equations (17e) and (17c). Equation (26b) could have been obtained from equations (17b) by neglecting the dependence on Φ.

Two-dimensional temperature field

When the temperature distribution in a cylindrical medium is a function of r and Φ, and independent of z, the radiation intensity is a function of the position vector $\mathbf{R} = r\mathbf{e}_r$ and the directional vector $\mathbf{e}_\Omega$. The radiation transport equation and boundary condition are independent of z, i.e.

$$\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\Phi} - \frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\phi} + \alpha_a\right)I = \alpha_a\sigma T_g^4(r,\Phi)/\pi, \tag{27}$$

$$I(r^*,\Phi^*,\phi^*,\theta^*) = \frac{\sigma T_w^4}{\pi}(r^*,\Phi^*), \tag{28}$$

where r^*, Φ^*, ϕ^*, and θ^* are specified on the wall.

To obtain formal solution of equation (27) with boundary condition (28), it is convenient to recast these equations in terms of the new variables s_1 and s_2 where $s_1 = \mathbf{R}.\mathbf{e}_\rho$ and $s_2 = \mathbf{R}.\mathbf{e}_\phi$ in which $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are unit vectors of the local polar coordinates given by

$$\mathbf{e}_\rho = \cos\phi\,\mathbf{e}_r + \sin\phi\,\mathbf{e}_\Phi, \qquad \mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_r + \cos\phi\,\mathbf{e}_\Phi. \tag{29}$$

It follows that

$$s_1 = \mathbf{R}.\mathbf{e}_\rho = r\cos\phi, \qquad s_2 = \mathbf{R}.\mathbf{e}_\phi = -r\sin\phi. \tag{30}$$

Equations (27) and (28) in terms of the new variables are

$$\left(\frac{\partial}{\partial s_1} + \alpha_a'\right)I' = \alpha_a'\sigma T_g^4[s_1, s_2, \phi]/\pi, \tag{31}$$

and

$$I'(s_1^*, s_2^*, \phi^*) = \frac{\sigma T_w^4}{\pi}(s_1^*, s_2^*, \phi^*), \tag{32}$$

where $s_1^* = r^*\cos\phi^*$, $s_2^* = -r^*\sin\phi^*$, $\alpha_a' = \alpha_a/\sin\theta$, and $I'(r,\Phi,\phi) = I(r,\Phi,\phi,\theta)$.

If the absorption coefficient is constant, the formal solution to equation (31), when written in the original variables I, is given by equation (3). To determine C, we note that equations (13b) and (13c) also hold for the two-dimensional situation, which in turn yields equations (14a) and (14b). When the solution for I is obtained, the radiative quantities are determined from

$$M(r,\Phi) = \int_\Omega I(r,\Phi,\phi,\theta)\,\mathrm{d}\Omega, \quad q_r(r,\Phi) = \int_\Omega I(r,\Phi,\phi,\theta)l_r\,\mathrm{d}\Omega, \quad q_\Phi(r,\Phi) = \int_\Omega I(r,\Phi,\phi,\theta)l_\Phi\,\mathrm{d}\Omega, \tag{33}$$

and $q_z(r,\Phi) = 0$. We shall now consider some specific problems in a two-dimensional temperature field.

Emitting–absorbing medium inside an infinite cylinder. Consider an emitting–absorbing medium with temperature distribution $T_g(r,\Phi)$ inside an infinite cylinder with wall temperatures at $r = r_o$ given by $T_{o+}(r_o,\Phi^*)$ for $0 < \Phi^* \leqslant \Phi_2$ and $T_{o-}(r_o,\Phi^*)$ for $\Phi_1 \leqslant \Phi^* < 0$ and zero elsewhere. By imposing boundary conditions, it can be shown that the radiative quantities resulting from medium emission are given by

$$\begin{Bmatrix} M_g(r,\Phi) \\ q_{gr}(r,\Phi) \\ q_{g\Phi}(r,\Phi) \end{Bmatrix} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_g(r,\Phi,\phi,\theta)\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \end{Bmatrix} \mathrm{d}\Omega, \tag{34a}$$

where

$$I_g = \frac{\sigma}{\pi}\int_0^{[r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}]/\sin\theta} \alpha_a T_g^4(\bar{s}_1', s_2, \phi, \theta)\exp[\alpha_a(\bar{s}_1'-\bar{s}_1)]\,\mathrm{d}\bar{s}_1', \tag{34b}$$

whereas those resulting from wall emission are given by

$$\begin{Bmatrix} M_w(r,\Phi) \\ q_{wr}(r,\Phi) \\ q_{w\Phi}(r,\Phi) \end{Bmatrix} = \int_{\phi_A}^{\phi_B}\int_{\theta=0}^{\pi} I_w^{(o+)}\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \end{Bmatrix} \mathrm{d}\Omega + \int_{\phi_B}^{\phi_C}\int_{\theta=0}^{\pi} I_w^{(o-)}\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \end{Bmatrix} \mathrm{d}\Omega, \tag{35a}$$

where

$$I_w^{(o\pm)} = \frac{\sigma T_{o\pm}^4}{\pi}[r_o, \Phi^*]\exp\left[-\frac{\alpha_a(r\cos\phi+\sqrt{[r_o^2-r^2\sin^2\phi]})}{\sin\theta}\right], \tag{35b}$$

with $\Phi^* \equiv \Phi+\phi-\sin^{-1}(r/r_o\sin\phi)$ and ϕ_A, ϕ_B, and ϕ_C given by equations (22b–22d).

Equation (35b) could have been obtained directly from the general three-dimensional case equation (17b) by neglecting z^* because the wall contribution in this case remains unaltered no matter at what z^*, $\mathbf{e}_\Omega$ vector projected backwards meets the wall. The limits θ_B and θ_C as given by equations (17f) and (17g) reduced to 0 and π when $c_1 \to -\infty$ and $c_2 \to \infty$.

One-dimensional temperature field

When the temperature distribution is a function of r alone, the radiation intensity is a function of the position vector $\mathbf{R} = r\mathbf{e}_r$ and the directional vector $\mathbf{e}_\Omega$. The radiation-transport equation and boundary conditions are given by

$$\left(\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\phi} + \alpha_a\right)I = \frac{\alpha_a\sigma T_g^4}{\pi}(r), \tag{36}$$

and

$$I(r^*,\phi^*,\theta^*) = \frac{\sigma T_w^4}{\pi}(r^*). \tag{37}$$

It can be shown that equations (36) and (37) in terms of the new variables s_1 and s_2 given by equations (30), reduce to equations with the same form as equations (31) and (32), which have the formal solution given by equation (3). To determine C, we note that

$$s_2 = s_2^*, \tag{38}$$

which gives

$$\cos\phi^* = \pm\frac{1}{r^*}\sqrt{(r^{*2}-r^2\sin^2\phi)}. \tag{39}$$

We shall now consider the cases of emitting–absorbing medium inside an infinite cylinder and inside and outside of an infinite concentric cylinder where temperature distribution is a function of r and independent of Φ and z.

Emitting–absorbing medium inside an infinite cylinder. Consider an emitting–absorbing medium with temperature $T_g(r)$ in an infinite cylinder with wall temperature given by T_o at $r = r_o$. Imposing the boundary condition gives

$$I_g(r,\phi,\theta) = \frac{\sigma}{\pi}\int_0^{[r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)}]/\sin\theta} \alpha_a T_g^4(\bar{s}_1', s_2, \theta, \phi)\exp[\alpha_a(\bar{s}_1'-\bar{s}_1)]\,\mathrm{d}\bar{s}_1', \tag{40a}$$

$$I_w(r,\phi,\theta) = \frac{\sigma T_o^4}{\pi}\exp\left[-\frac{\alpha_a(r\cos\phi+\sqrt{(r_o^2-r^2\sin^2\phi)})}{\sin\theta}\right]. \tag{40b}$$

The radiative quantities resulting from medium emission are

$$\begin{Bmatrix} M_g(r) \\ q_{gr}(r) \end{Bmatrix} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_g(r,\phi,\theta) \begin{Bmatrix} 1 \\ l_r \end{Bmatrix} d\Omega, \tag{41a}$$

and those resulting from wall emission are

$$\begin{Bmatrix} M_w(r) \\ q_{wr}(r) \end{Bmatrix} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_w(r,\phi,\theta) \begin{Bmatrix} 1 \\ l_r \end{Bmatrix} d\Omega. \tag{41b}$$

Equation (40b) for a one-dimensional case could have been obtained directly from the two-dimensional case by neglecting Φ^* or from the three-dimensional case by neglecting both Φ^* and z^* and by letting $c_1 \to -\infty$ and $c_2 \to \infty$; thereby finding the appropriate limits for θ as being from 0 to π. The reason for neglecting Φ^* and z^* is because the wall contribution in this case remains unaltered, no matter at which Φ^* and z^*, $\mathbf{e}_\Omega$ vector projected back meets the wall.

We shall now prove that equation (40) is indeed equivalent to a much more complicated solution obtained by Kestin [2] who gave two separate expressions for the radiation intensity for $0 \leqslant x < r_o \cos\beta$ and $r_o \cos\beta < x \leqslant 2r_o \cos\beta$. The complication of Kestin's solution is owing to the particular coordinate system chosen for the problem. To show the equivalence of the two solutions, we note that the angles α and β in Kestin's paper are related to θ and ϕ in the present work by the relations

$$\alpha = \frac{\pi}{2} - \theta, \tag{42a}$$

$$\sin\beta = \frac{r}{r_o} \sin\phi. \tag{42b}$$

With the aid of equations (42), the integral

$$\int_{r_o}^{r} \frac{dr'}{F(r',\beta)}$$

in Kestin's paper can be integrated to give

$$\int_{r_o}^{r} \frac{dr'}{F(r',\beta)} = -\alpha_a \int_{r_o}^{r} \frac{r\,dr'}{\sqrt{(r^2 - r_o^2 \sin\beta)}} = -\alpha_a (r\cos\phi + \sqrt{[r_o^2 - r^2 \sin\phi]}). \tag{43}$$

Thus our expression for wall emission is identical to that obtained by Kestin. For the medium emission, we note that the relation $dx = r\,dr/[r^2 - r_o^2 \sin^2\beta]^{1/2}$ in Kestin's paper can be shown to be equivalent to

$$dx = -\frac{dr}{\alpha_a F(r,\beta)} = \frac{dr}{\cos\phi}. \tag{44}$$

Furthermore, since $s_1 = r\cos\phi$, and $s_2 = r\sin\phi$ = constant, the differentiation of s_1 and s_2 gives

$$ds_1 = \frac{dr}{\cos\phi}. \tag{45}$$

Equations (44) and (45) establish the identity between dx and ds_1 in two papers under comparison. With the aid of these equations, it is now apparent that Kestin's expression for medium emission $(0 \leqslant x < r_o \cos\beta)$

$$I_g(r,\beta,\mu) = -\frac{\exp\left[\frac{1}{\mu}\int_{r_o}^{r} \frac{dr'}{F(r',\beta)}\right]}{\mu} \int_{r_o}^{r} \frac{\exp\left[-\frac{1}{\mu}\int_{r_o}^{r} \frac{dr'}{F(r',\beta)}\right] \frac{\sigma T_g^4}{\pi}(r')}{F(r',\beta)} dr', \tag{46}$$

with $\mu \equiv \cos\alpha$

can be reduced to our expression given by equation (40a).

Emitting–absorbing medium between infinite concentric cylinders. Consider an emitting–absorbing medium with axial symmetric temperature distribution between infinite concentric cylinders with outer and inner wall temperatures given by $T_o(r_o)$ and $T_i(r_i)$ respectively. By imposing the boundary conditions, it can be shown that the radiative quantities resulting from medium emission are given by

$$\begin{Bmatrix} M_g(r) \\ q_{gr}(r) \end{Bmatrix} = \int_{-\phi_0}^{\phi_0} \int_{0}^{\pi} I_g^{(i)}(r,\phi,\theta) \begin{Bmatrix} 1 \\ l_r \end{Bmatrix} d\Omega + \int_{\phi_0}^{2\pi-\phi_0} \int_{0}^{\pi} I_g^{(o)}(r,\phi,\theta) \begin{Bmatrix} 1 \\ l_r \end{Bmatrix} d\Omega, \tag{47}$$

where

$$I_g^{(i)} = \frac{\sigma}{\pi}\int_0^{[r\cos\phi - \sqrt{(r_i^2 - r^2\sin^2\phi)}]/\sin\theta} \alpha_a T_g^4(\tilde{s}_1', s_2, \phi, \theta)\exp[\alpha_a(\tilde{s}_1' - \tilde{s}_1)]\,\mathrm{d}\tilde{s}_1', \tag{48a}$$

$$I_g^{(o)} = \frac{\sigma}{\pi}\int_0^{[r\cos\phi + \sqrt{(r_o^2 - r^2\sin^2\phi)}]/\sin\theta} \alpha_a T_g^4(\tilde{s}_1', s_2, \phi, \theta)\exp[\alpha_a(\tilde{s}_1' - \tilde{s}_1)]\,\mathrm{d}\tilde{s}_1', \tag{48b}$$

and those resulting from wall emission are also given by equation (47) with $I_g^{(o)}$ and $I_g^{(i)}$ replaced by $I_w^{(o)}$ and $I_w^{(i)}$ which are given by equations (26) by dropping the dependence on z^*.

Emitting–absorbing medium outside of an infinite cylinder

For the case of an emitting–absorbing medium with temperature distribution $T_g(r)$ outside of an infinite cylinder, solution can be obtained from equation (47) by letting $r_o \to \infty$ and $T_o = 0$.

RESULTS AND DISCUSSIONS

Numerical results are obtained for special cases of isothermal media inside cylinders and concentric cylinders with finite and infinite length, bounded by isothermal and piecewise isothermal walls. In these special cases the expressions for radiative quantities given in the previous section can be simplified considerably. For example, the radiative quantities in an isothermal medium inside a finite cylinder bounded by isothermal walls at $z = \pm c$, and piecewise isothermal wall at $r = r_o$ given by equations (20) and (22) can be simplified to give

$$\begin{Bmatrix} 4\tilde{M}_g(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{gr}(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{g\Phi}(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{gz}(\tilde{r},\Phi,\tilde{z}) \end{Bmatrix} = \frac{\tilde{T}_g^4}{\pi}\int_{\phi=0}^{2\pi}\int_0^{\theta_B}\left\{1-\exp\left[-\frac{Bu(\tilde{z}+\tilde{c})}{\cos\theta}\right]\right\}\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega$$

$$+\frac{\tilde{T}_g^4}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta_B}^{\theta_C}\left\{1-\exp\left[-\frac{Bu\left(\tilde{r}\cos\phi+\sqrt{(\tilde{r}_o^2-\tilde{r}^2\sin^2\phi)}\right)}{\sin\theta}\right]\right\}\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega$$

$$+\frac{\tilde{T}_g^4}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta_C}^{\pi}\left\{1-\exp\left[-\frac{Bu(\tilde{z}-\tilde{c})}{\cos\theta}\right]\right\}\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega, \tag{49a}$$

and

$$\begin{Bmatrix} 4\tilde{M}_w(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{wr}(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{w\Phi}(\tilde{r},\Phi,\tilde{z}) \\ \tilde{Q}_{wz}(\tilde{r},\Phi,\tilde{z}) \end{Bmatrix} = \frac{\tilde{T}_1^4}{\pi}\int_{\phi=0}^{2\pi}\int_0^{\theta_B}\exp\left[-\frac{Bu(\tilde{z}+\tilde{c})}{\cos\theta}\right]\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega$$

$$+\frac{\tilde{T}_{o-}^4}{\pi}\int_{\phi_A}^{\phi_B}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{Bu\left(\tilde{r}\cos\phi+\sqrt{(\tilde{r}_o^2-\tilde{r}^2\sin^2\phi)}\right)}{\sin\theta}\right]\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega$$

$$+\frac{\tilde{T}_{o+}^4}{\pi}\int_{\phi_B}^{\phi_C}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{Bu\left(\tilde{r}\cos\phi+\sqrt{(\tilde{r}_o^2-\tilde{r}^2\sin^2\phi)}\right)}{\sin\theta}\right]\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega$$

$$+\frac{\tilde{T}_2^4}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta_C}^{\pi}\exp\left[-\frac{Bu(\tilde{z}-\tilde{c})}{\cos\theta}\right]\begin{Bmatrix} 1 \\ l_r \\ l_\Phi \\ l_z \end{Bmatrix}\mathrm{d}\Omega, \tag{49b}$$

where $\tilde{M} = M/4\sigma T_o^4$, $\tilde{Q} = q/\sigma T_o^4$, $\tilde{T}_g = T_g/T_o$, $\tilde{r} = r/L$, $\tilde{z} = z/L$, $\tilde{c} = c/L$ and $Bu = \alpha_a L$, with T_o and L denoting the reference temperature and reference length. For simplicity of presentation and discussion, we shall henceforth omit the "$\tilde{}$" and all radiative quantities are understood to be normalized variables. For the special case of $T_g = T_{o\pm} = T_1 = T_2 = 1$, equations (49a) and (49b) yield

$$M_g + M_w = 1,\quad Q_{gr} + Q_{wr} = 0,\quad Q_{g\Phi} + Q_{w\Phi} = 0,\quad Q_{gz} + Q_{wz} = 0. \tag{50}$$

Relations analogous to equations (50) are valid whenever medium and walls are maintained at the same temperature.

Figure 2 shows the comparison of radiation fields along the radial direction at plane $z = 0$, for an isothermal medium ($T_g = 1$) enclosed by a finite ($r_o = c = 1$) as well as an infinite ($r_o = 1$ and $c \to \infty$) cylinder with isothermal walls ($T_w = 1$). A similar comparison for finite and infinite concentric cylinders ($r_o = 1, r_i = 0{\cdot}5$) is shown in Fig. 3. In both of these plots, it is observed that for a specific value of Bu, the value of M_g as well as Q_{g_r} $(= -Q_{w_r})$ increases as the length of cylinder is increased, (ii) the effect of length, c, however, is negligible for $Bu = 5{\cdot}0$ (i.e. for an optically thick gas) indicating, thereby, that a three-dimensional problem can be approximated by a two-dimensional one without any significant loss of accuracy under these circumstances, (iii) the boundary-layer behavior near the walls becomes more pronounced as the magnitude of Bu is increased; this phenomenon can be explained on account of the exponential damping effect: that is, for large Bu, the wall effects are felt only in the region near the walls, (iv) for small values of Bu, $M_w \gg M_g$ and $Q_{g_r}(= -Q_{w_r})$ is small everywhere. The detailed variation of radiative quantities along the radius is, however, different for the case of a cylinder from that of a concentric cylinder. For the case of an isothermal

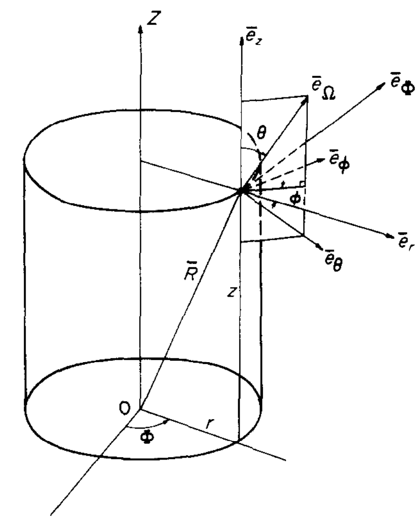

FIG. 1. Cylindrical coordinate system.

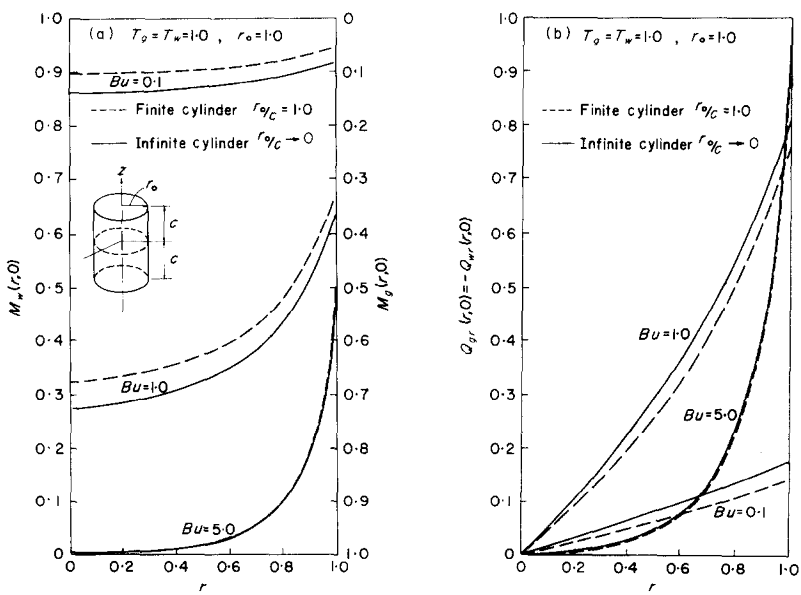

FIG. 2. Comparison of radiation fields along (r, o) in an isothermal medium inside finite and infinite cylinders with isothermal walls.

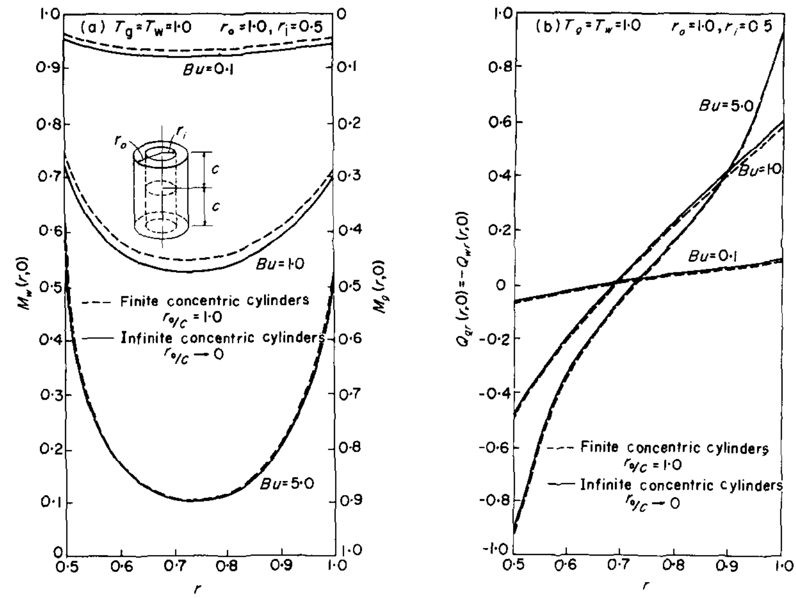

FIG. 3. Comparison of radiation fields along (r, o) in an isothermal medium inside finite and infinite concentric cylinders with isothermal walls.

medium inside a cylinder (Fig. 2), M_g is maximum at the center ($r = 0$) and decreases away from the center whereas Q_{g_r} is zero at the center and increases to a maximum at the wall ($r = 1$). In case of an isothermal medium between concentric cylinders (Fig. 3). M_g is a maximum at a point near halfway between inner and outer cylinders. The value of Q_{g_r} is negative on the surface of inner cylinder, and positive on the surface of outer cylinder. This result is intuitively anticipated because the net heat flux in the r-direction from the hot medium, enclosed by inner and outer cylinders at zero wall temperature, will cross the inner surface in the negative direction of r, whereas it will cross the outer surface in the positive direction of r.

Figure 4 shows the comparison of radiative quantities along z-direction on the wall at $r = 1$, for an isothermal medium enclosed in an isothermal cylinder as well as isothermal concentric cylinders with finite length. It is shown that the variation of radiative quantities along $(1, z)$ in Figs. 4(a) and (b) has qualitatively the same shape as those along (r, o) in Figs. 2(a) and (b). In Fig. 4 it is shown that values of

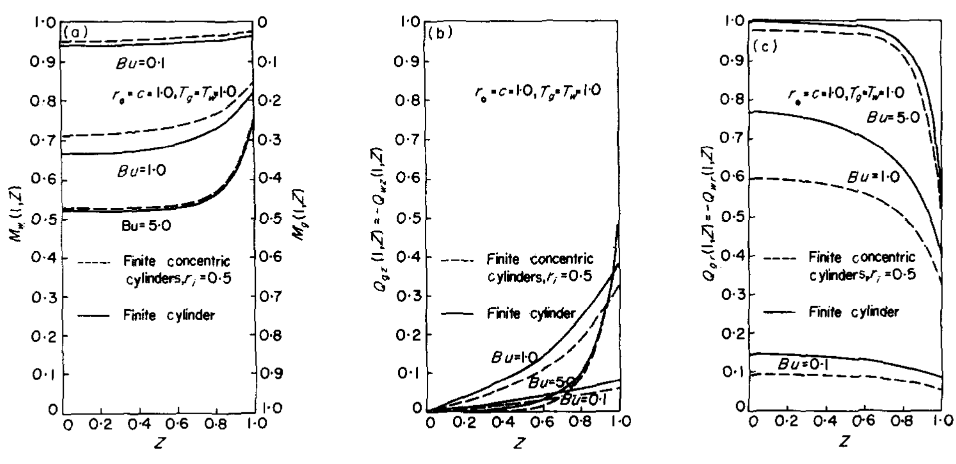

FIG. 4. Comparison of radiation fields along $(1, z)$ in an isothermal medium inside cylinder and concentric cylinders with finite length bounded by isothermal walls.

M_g and Q_{g_r} $(= -Q_{w_r})$ in case of a cylinder are higher than those between concentric cylinders. Also it is observed that M_g and Q_g have maximum values at $z = 0$ and the same decrease as $z = 1$ is approached. This is also a well anticipated result and is explainable by the fact that at $z = 1$, the radiative quantities associated with wall become more predominating than those associated with the medium due to the effect of emitting plates placed at $z = \pm 1$.

The effects of discontinuous wall temperature at $r = 1$ on radiative quantities resulting from wall emission in a finite cylinder are shown in Figs. 5 and 6. Figure 5 shows the case where wall temperature is discontinuous at $\Phi = 0$ and $\Phi = \pi$ (with $T_{o-} = 1{\cdot}1$ for $\pi < \Phi < 2\pi$ and $T_{o+} = 1{\cdot}2$ for $0 < \Phi < \pi$) whereas Fig. 6 shows the case where wall temperature is discontinuous at $z = 0$ (with $T_{c-} = 1{\cdot}1$ for $-1 < z < 0$ and $T_{c+} = 1{\cdot}2$ for $0 < z < 1$). It is observed from these plots that: (i) at the point where wall temperature is discontinuous, M_w has a jump in value due to differential wall temperatures across the point of discontinuity; this jump depends only on the wall temperatures and is independent of Bu, (ii) whereas the heat flux component is continuous in the direction in which wall temperature is discontinuous, heat flux components in all other directions are discontinuous. For example, Fig. 5 shows the case where wall temperature at $r = 1$ is discontinuous in Φ direction. It is observed in this plot that Q_{w_Φ} is continuous but Q_{w_r} is discontinuous at $\Phi = 0$. Similarly, Fig. 6 shows

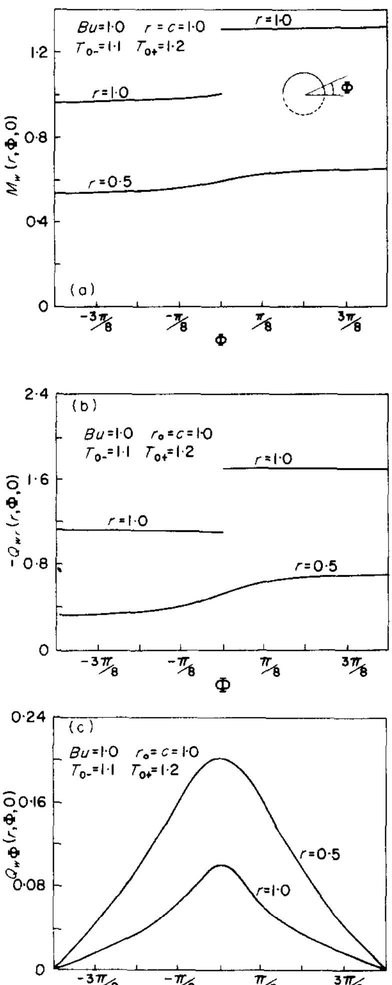

FIG. 5. Effects of discontinuous wall temperature in Φ-direction on the radiation field inside a finite cylinder.

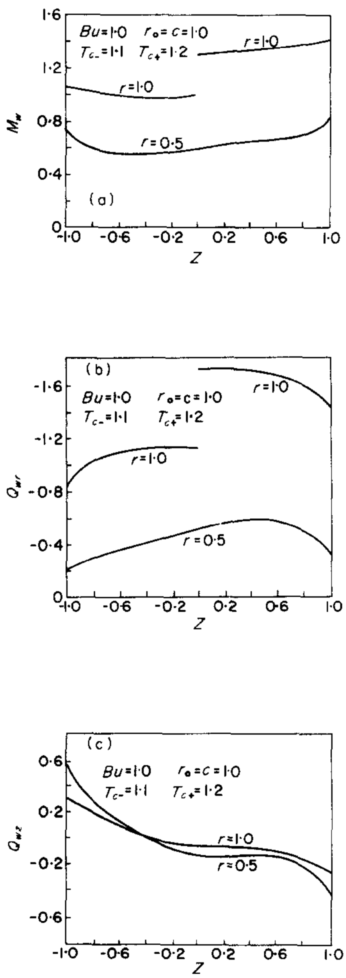

FIG. 6. Effects of discontinuous wall temperature in z-direction on the radiation field inside a finite cylinder.

the case where wall temperature is discontinuous in z-direction. It is shown that Q_{w_z} is continuous whereas Q_{w_r} is discontinuous at $z = 0$. (iii) Away from the wall ($r = 0{\cdot}5$ for example), all radiative quantities are continuous.

The first two observations can be shown analytically as follows. Consider the case where wall temperature (at $r = r_o$) is discontinuous at $\Phi = 0$, and $\Phi = \pi$, as indicated in Fig. 5. On the wall at $r = r_o$, equation (49b) can be rewritten as

(i) for $0 < \Phi < \pi$

$$\begin{Bmatrix} 4\tilde{M}_w(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{w_r}(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{w\Phi}(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{wz}(\tilde{r}_o,\Phi,\tilde{z}) \end{Bmatrix}$$

$$= \frac{\tilde{T}_1^4}{\pi}\int_{\phi=0}^{2\pi}\int_0^{\theta_B}\exp\left[-\frac{Bu(\tilde{z}+\tilde{c})}{\cos\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o-}^4}{\pi}\int_{\phi_A}^{\phi_B}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o+}^4}{\pi}\int_{\phi_B}^{\pi/2}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o+}^4}{\pi}\int_{\pi/2}^{3\pi/2}\int_{\theta_B}^{\theta_C}\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o+}^4}{\pi}\int_{3\pi/2}^{\phi_A}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_2^4}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta_C}^{\pi}\exp\left[-\frac{Bu(\tilde{z}-\tilde{c})}{\cos\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega. \quad (51a)$$

(ii) for $\pi < \Phi < 2\pi$

$$\begin{Bmatrix} 4\tilde{M}_w(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{wr}(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{w\Phi}(\tilde{r}_o,\Phi,\tilde{z}) \\ \tilde{Q}_{wz}(\tilde{r}_o,\Phi,\tilde{z}) \end{Bmatrix}$$

$$= \frac{\tilde{T}_1^4}{\pi}\int_{\phi=0}^{2\pi}\int_0^{\theta_B}\exp\left[-\frac{Bu(\tilde{z}+\tilde{c})}{\cos\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o-}^4}{\pi}\int_{\phi_A}^{\pi/2}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o-}^4}{\pi}\int_{\pi/2}^{3\pi/2}\int_{\theta_B}^{\theta_C}\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o-}^4}{\pi}\int_{3\pi/2}^{\phi_B}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_{o+}^4}{\pi}\int_{\phi_B}^{\phi_A}\int_{\theta_B}^{\theta_C}\exp\left[-\frac{2Bu\,\tilde{r}_o\cos\phi}{\sin\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega$$

$$+\frac{\tilde{T}_2^4}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta_C}^{\pi}\exp\left[-\frac{Bu(\tilde{z}-\tilde{c})}{\cos\theta}\right]\begin{Bmatrix}1\\ l_r\\ l_\Phi\\ l_z\end{Bmatrix}d\Omega. \quad (51b)$$

The first and the last integrals in equations (51a) and (51b) account for the isothermal wall emission from the top and bottom plates at $z = \pm c$. The remaining integrals represent the wall emission at $r = r_o$; the physical meaning of these integrals can be easily seen in Fig. 7. The second, third, and fifth integrals in equation (51a) represent incident radiation at the point P while the fourth integral represents the local wall emission. Similarly, the second, fourth and fifth integrals represent incident radiation at the point M

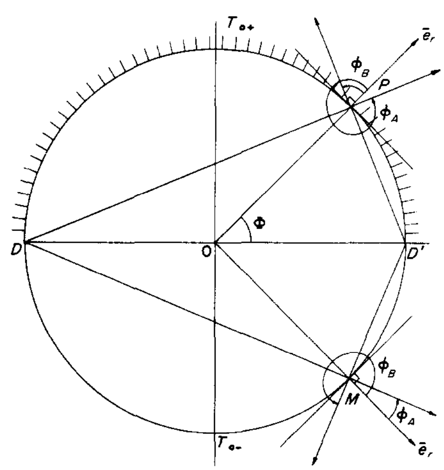

FIG. 7. Physical significance of the integrals in equations (51a) and (51b).

with the third integral representing the local wall emission at the same point. Consider the points at $\Phi = o+$ and $\Phi = o-$ on the wall ($r = 1$) and at $z = 0$ (see Fig. 7). It can be shown from equations (51a) and (51b) that at these points incident radiation are identical, and that the radiative quantities, if they are different at all, are due to the local wall emission which in turn depends on the local wall temperature. Thus the jump in values at $\Phi = 0$ are given by the difference resulting from local wall emission, i.e.

$$\begin{Bmatrix} 4\Delta M_w \\ \Delta Q_{wr} \\ \Delta Q_{w\Phi} \end{Bmatrix} = \left[\frac{(1{\cdot}2)^4 - (1{\cdot}1)^4}{\pi}\right] \int_{\phi=\pi/2}^{3\pi/2} \int_{\theta=0}^{\pi} \begin{Bmatrix} 1 \\ l_r \\ l_\Phi \end{Bmatrix} d\Omega = \begin{Bmatrix} 1{\cdot}22 \\ 0{\cdot}61 \\ 0 \end{Bmatrix}.$$

These values are indeed confirmed by the numerical results.

The above argument is only valid at the wall. Away from wall there is no local wall emission, and radiation from all points on the wall are incident radiation. Since any point in the radiation field "sees" all parts of the walls, the radiative quantities are continuous away from the wall. It should be noted that the value of $Q_{w\Phi}$ in Fig. 5(c) is almost symmetrical about $\Phi = 0$ at $Bu = 1$. It is expected that for higher values of Bu, the unsymmetrical distribution of $Q_{w\Phi}$ will become apparent: because for higher values of Bu, most of incident radiation originated from the wall at a position further away, is absorbed by the optically thick medium before it can reach the point under consideration, and hence local emission predominates more at this point, and thus bring out the effect of discontinuous wall temperature on $Q_{w\Phi}$ more clearly. Furthermore, since M_w is discontinuous along Φ (at $r = 1$) in Fig. 5, and is discontinuous along z (at $r = 1$) in Fig. 6, the derivatives $\partial M_w/\partial\Phi$ in Fig. 5 and $\partial M_w/\partial z$ in Fig. 6 are not, therefore, unique at $r = 1$. Consequently, differential approximation cannot be applied in such a case [15].

CONCLUDING REMARKS

In this paper we have obtained the exact expressions for multi-dimensional radiative heat flux in cylindrical coordinate system, in terms of its temperature distribution, explicitly. With this expression for radiative heat flux, exact formulation of multi-dimensional radiation-coupled energy-transport problems in cylindrical coordinate is now possible. The result will lead to a set of integro-differential equations which will require numerical solutions.

The key to the success of the present approach in obtaining exact expressions for multi-dimensional radiative heat flux, in terms of temperature distribution, lies in the identification of the coordinate transformation, equation (2), the relations [14] for a three-dimensional temperature field, and the corresponding equations for a two-dimensional temperature field. These relations are different from the corresponding relations for the rectangular coordinate system presented in [15].

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TRANSFERT MULTIDIMENSIONNEL PAR RAYONNEMENT DANS UN MILIEU CYLINDRIQUE NON ISOTHERME LIMITE PAR DES PAROIS NON ISOTHERMES

Résumé—On décrit une approche systématique du calcul exact du flux thermique multidimensionnel par rayonnement dans un milieu non isotherme qui émet et absorbe, avec des parois non isothermes. Des solutions exactes finies sont obtenues pour le transfert à l'intérieur de cylindres (finis ou infinis) et de cylindres concentriques avec une distribution de température tridimensionnelle. Dans le cas spécial d'une situation axisymétrique et monodimensionnelle, la solution exacte obtenue est équivalente à une solution exacte déja obtenue dans une autre étude, bien que celle qui est présentée ici soit plus élégante dans l'approche et de forme plus simple. On présente aussi des résultats numériques dans le cas d'un milieu isotherme limité par des parois isothermes par morceaux.

MEHRDIMENSIONALER STRAHLUNGSAUSTAUSCH IN NICHT ISOTHERMEN, ZYLINDRISCHEN STOFFEN MIT NICHT ISOTHERMEN GRENZFLÄCHEN

Zusammenfassung—Es wird eine systematische Näherung für die exakte Berechnung des mehrdimensionalen Strahlungs-Wärmestroms in einem zylindrischen, emittierenden und absorbierenden, nicht isothermen Stoff mit nicht isothermen Grenzflächen beschrieben. Man erhält exakte Lösungen in geschlossener Form für den Strahlungsaustausch innerhalb (endlicher und unendlicher) Zylinder und konzentrischer Zylinder mit vorgegebener dreidimensionaler Temperaturverteilung. Für den speziellen Fall der eindimensionalen zylindrischen Symmetrie wird gezeigt, daß die exakte Lösung, die in der vorliegenden Arbeit gewonnen wird, der exakten Lösung entspricht, die in einer früheren Arbeit angegben wurde, obgleich die vorliegende Lösung viel eleganter in der Näherung und einfacher in der Form ist. Für den Fall des isothermen Stoffes, der durch teilweise isotherme Wände begrenzt ist, werden auch Zahlenwerte angegeben.

МНОГОМЕРНЫЙ ЛУЧИСТЫЙ ТЕПЛООБМЕН В НЕИЗОТЕРМИЧЕСКОЙ ЦИЛИНДРИЧЕСКОЙ СРЕДЕ С НЕИЗОТЕРМИЧЕСКИМИ СТЕНКАМИ

Аннотация — В работе описывается метод точного расчета многомерного лучистого теплового потока в цилиндрической излучающе-поглощающей неизотермической среде с неизотермическими стенками. Получены точные решения в замкнутой форме для лучистого переноса внутри конечных и бесконечных цилиндров, а также в концентрических цилиндрах с заданным трехмерным распределением температуры. Показано, что для частного случая одномерного симметричного цилиндра точное решение, полученное в этом исследовании, эквивалентно точному решению предыдущей работы, хотя настоящее решение является более элегантным и простым по форме. Представлены также численные результаты для изотермической среды, ограниченной кусочными изотермическими стенками.